GAUGING THE WESS–ZUMINO TERM OF A SIGMA MODEL WITH BOUNDARY

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Abstract. We investigate the gauging of the Wess–Zumino term of a sigma model with boundary. We derive a set of obstructions to gauging and we interpret them as the conditions for the Wess–Zumino term to extend to a closed form in a suitable equivariant relative de Rham complex. We illustrate this with the two-dimensional sigma model and we show that the new obstructions due to the boundary can be interpreted in terms of Courant algebroids. We specialise to the case of the Wess–Zumino–Witten model, where it is proved that there always exist suitable boundary conditions which allow gauging any subgroup which can be gauged in the absence of a boundary. We illustrate this with two natural classes of gaugings: (twisted) diagonal subgroups with boundary conditions given by (twisted) conjugacy classes, and chiral isotropic subgroups with boundary conditions given by cosets.

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1. Introduction

Ever since the advent of D-branes, field theories on manifolds with boundary have attracted a great deal of attention. One such theory, with fascinating connections to geometry and topology, is the nonlinear sigma model and in particular, in the context of string theory, the two-dimensional nonlinear sigma model. The sigma model describes harmonic maps between two (pseudo)riemannian manifolds, which we will call the spacetime and the target space. Isometries

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of the target space become classical global symmetries of the sigma model and as with all classical symmetries we can try to gauge them, that is, promote them to local symmetries by coupling the model to gauge fields. For a standard sigma model, whether or not the spacetime has boundary, there is no obstruction in doing so; however things are different when there is a (topologically nontrivial) Wess–Zumino (WZ) term.

The gauging of the WZ term of a nonlinear sigma model was the subject of much work in the late 1980s and early 1990s, spurred on by the pivotal role played by the Wess–Zumino–Witten (WZW) model [1] in a number of areas: string theory, conformal field theory and integrable systems. The two-dimensional case was investigated independently in [2, 3] and later generalised to higher dimension in [4]. As the dimension grows the number of obstructions increases and a very clear pattern emerges. This was interpreted in terms of equivariant cohomology independently in [5, 6, 7]; although see also [8] for the gauging of the WZW model.

The purpose of this note is to re-examine this problem when the spacetime has boundary. The WZ term of a sigma model with boundary is a straight-forward generalisation of the case of the boundary WZW model treated in [9, 10, 11] and will be reviewed below. We extend the interpretation of gauging the WZ term in terms of a relative version of equivariant cohomology and in this way derive a number of new obstructions. Remarkably, for the case of the WZW model, these new obstructions are automatically overcome and we will show that we can gauge *any* subgroup which can be gauged in the absence of a boundary, provided that the boundary conditions are chosen appropriately. This extends some recent results [12, 13, 14] on gauged WZW models with boundary conditions given by (twisted) conjugacy classes. For the case of the two-dimensional sigma model we interpret the new obstructions in terms of the existence of subalgebroids of the Courant algebroid of the target and of its restriction to the boundary submanifold.

This note is organised as follows. In Section 2 we recall the definition of the WZ term in a sigma model with boundary. We recall the topological conditions needed for the physical consistency of such a term. In Section 3 we recall the obstructions to gauging a WZ term in the case of a sigma model without boundary and their interpretation in terms of equivariant cohomology and in particular their derivation using the Cartan model for the equivariant cohomology. In Section 4 we derive the obstructions to gauging the WZ term in a sigma model with boundary. This is derived using the Cartan model for what could be termed relative equivariant cohomology. We are aware of two competing complexes which could claim to compute the relative equivariant cohomology and we explain this situation in Appendix A. We illustrate our results throughout with the case of a two-dimensional sigma model. We also re-interpret the conditions for gauging in terms of Courant algebroids, extending this observation to arbitrary dimension and to the presence of boundary. Finally, in Section 5 we apply these considerations to the WZW model and show that we may gauge *any* subgroup which can be gauged in the absence of boundary, provided we choose compatible boundary conditions, namely orbits of the action of the group we are trying to gauge. We illustrate this with two natural classes of gaugings: twisted diagonal gaugings with boundary conditions given by twisted conjugacy classes, and chiral isotropic gaugings with boundary conditions given by cosets.

2. The WZ term of a sigma model with boundary

In this section we review the definition of the WZ term for a sigma-model with boundary and the conditions of the well-definedness of the path integral. We follow the treatment in [11], to where the reader is referred for more details.

Let Σ be an oriented riemannian d-dimensional manifold and X be an oriented pseudo-riemannian manifold with $\dim X = n$. Let us first assume that Σ has no boundary. The sigma model is a theory of maps $\varphi : \Sigma \to X$. Let g denote the metric of X, η the metric on Σ and $\star$ the Hodge star operation on Σ. Relative to a local coordinate chart x^i on X and letting $\varphi^i = x^i \circ \varphi$ denote the component functions, the sigma model action takes the form

$$S_\sigma[\varphi] := \tfrac{1}{2} \int_\Sigma g_{ij}(\varphi) d\varphi^i \wedge \star d\varphi^j = \tfrac{1}{2} \int_\Sigma g_{ij}(\varphi) \partial_\alpha \varphi^i \partial_\beta \varphi^j \eta^{\alpha\beta}\, \mathrm{dvol}_\Sigma \ . \tag{1}$$

In more invariant terms, it is (one half of) the L^2 norm of $d\varphi \in \Omega^1(\Sigma, \varphi^* TX)$.

To this action it might be possible to add a WZ term. Let H be a closed $(d+1)$-form on X. Let $\varphi(\Sigma) \subset X$ denote the image of Σ under the map φ. Since Σ has no boundary, neither does its image. If there is a $(d+1)$-dimensional submanifold (or indeed a $(d+1)$-chain) $M \subset X$ whose boundary is $\varphi(\Sigma)$ then we can add a WZ term to the action

$$S_{\mathrm{WZ}} := \int_M H \ . \tag{2}$$

Since $dH = 0$ the variation of the WZ term is a boundary term

$$\delta S_{\mathrm{WZ}} = \int_M \mathscr{L}_{\delta\varphi} H = \int_M d\imath_{\delta\varphi} H = \int_{\varphi(\Sigma)} \imath_{\delta\varphi} H \ ,$$

whence its contribution to the equations of motion involve only the original map $\varphi : \Sigma \to X$.

The existence of M requires that the cycle $\varphi(\Sigma)$ bounds, whence its homology class $[\varphi(\Sigma)] \in H_d(X)$ vanishes. This has to be true for all maps φ and this is guaranteed by $H_d(X) = 0$. The archetypal example is the WZW model when $d = 2$ and X is a simply-connected compact simple Lie group, g a bi-invariant metric and H the standard bi-invariant three-form. It is a theorem of Cartan that $H_2(X) = 0$ in this case.

There is also the question of the dependence on M. As seen above, the classical equations of motion are independent of M, but the quantum theory in principle does depend on the choice of M. Indeed, suppose that $M' \subset X$ is another chain whose boundary is $\varphi(\Sigma)$. The difference between the WZ terms corresponding to M and M' is

$$\Delta S_{\mathrm{WZ}} = \int_M H - \int_{M'} H = \int_N H \ ,$$

where N is the cycle obtained by gluing M and M' (with the opposite orientation) along their common boundary $\varphi(\Sigma)$. Thus N defines a class $[N] \in H_{d+1}(X)$ and ΔS_{WZ} is the evaluation of the (de Rham) cohomology class $[H] \in H^{d+1}(X)$ on $[N]$. The path integral will be independent of the choice of M provided that $\Delta S_{\mathrm{WZ}} \in 2\pi\mathbb{Z}$, or equivalently, when $[H/2\pi]$ defines an integral cohomology class.

To summarise, there is an obstruction in $H_d(X)$ to defining the WZ term and a quantisation condition $\frac{1}{2\pi}[H] \in H^{d+1}(X;\mathbb{Z})$ to ensure that the (quantum) theory is well defined.

Now let us suppose that Σ has boundary $\partial\Sigma$. We specify boundary conditions by demanding that φ should map $\partial\Sigma$ to a given submanifold $Y \subset X$. We will abbreviate this by saying that $\varphi : (\Sigma, \partial\Sigma) \to (X, Y)$, suggesting a "relative" formulation of the problem. We can still write the sigma model term S_σ, but we immediately come to a problem with the WZ term: since Σ has boundary, so will its image $\varphi(\Sigma)$ and hence there will be no M with $\partial M = \varphi(\Sigma)$. The way out is to relativise the problem modulo Y; that is, we only demand that $\partial M = \varphi(\Sigma)$ *modulo* Y. In other words, we demand that $\partial M = \varphi(\Sigma) + D$ with some chain $D \subset Y$. The obstruction to the existence

of such $(M, D) \subset (X, Y)$ is the relative homology class $[\varphi(\Sigma)] \in H_d(X, Y)$. (See Appendix A for a brief scholium on relative (co)homology.)

Now suppose that there exists a d-form B on Y such that $i^*H = dB$, where $i : Y \hookrightarrow X$ denotes the embedding. This means that (H, B) defines a relative (de Rham) cohomology class $[(H, B)] \in H^{d+1}(X, Y)$. We can then write the following "relative" WZ term

$$S_{\mathrm{rWZ}} = \int_M H - \int_D B \ . \tag{3}$$

The d-form B does not enter in the equations of motion, but only in specifying the boundary conditions. Indeed, the variation of the relative WZ term is

$$\begin{aligned}\delta S_{\mathrm{rWZ}} &= \int_M \mathscr{L}_{\delta\varphi} H - \int_D \mathscr{L}_{\delta\varphi} B \\ &= \int_{\partial M} \imath_{\delta\varphi} H - \int_D \imath_{\delta\varphi} dB - \int_{\partial D} \imath_{\delta\varphi} B \\ &= \int_{\varphi(\Sigma)} \imath_{\delta\varphi} H + \int_{\varphi(\partial\Sigma)} \imath_{\delta\varphi} B \ ,\end{aligned}$$

where we have used that on D, $i^*H = dB$ and that $\partial D = -\partial\varphi(\Sigma) = -\varphi(\partial\Sigma)$.

There is also a quantisation condition which guarantees the independence of the path integral on the choice of (M, D). Suppose that $(M', D') \subset (X, Y)$ is such that $\partial M' = \varphi(\Sigma) + D'$. The difference in the WZ terms is now

$$\Delta S_{\mathrm{rWZ}} = \int_M H - \int_{M'} H - \int_D B + \int_{D'} B = \int_N H - \int_{\partial N} B \ ,$$

where $N = M - M'$ is the chain obtained by gluing M and M' (with the opposite orientation) along the common part of their boundary $\varphi(\Sigma)$. This still leaves some boundary ∂N which is easily calculated to be $D - D'$. This means that $(N, \partial N)$ defines a relative homology class $[(N, \partial N)] \in H_{d+1}(X, Y)$. The change in the WZ is then the evaluation of the relative (de Rham) cohomology class $[(H, B)]$ on $[(N, \partial N)]$ and the path integral will not see this provided that the result is in $2\pi\mathbb{Z}$, which implies the integrality of the relative cohomology class $[(H, B)]/2\pi$.

To summarise, there is an obstruction in $H_d(X, Y)$ to defining the WZ term and a quantisation condition $\frac{1}{2\pi}[(H, B)] \in H^{d+1}(X, Y; \mathbb{Z})$ to ensure that the theory is well defined. In other words, the situation is as in the case without boundary but now relative to Y.

3. Gauging the WZ term

In this section we review the gauging of the WZ term in a sigma model without boundary and its relation with equivariant cohomology. We follow the treatment in [7, 6].

We start with the setup up of the previous section, where Σ has no boundary. We assume throughout that G is a connected Lie group acting on X in such a way that both the metric g and H are preserved, so that the action $S_\sigma + S_{\mathrm{WZ}}$ is G-invariant. We can then set out to gauge the symmetry. Let $\mathfrak{g}$ denote the Lie algebra of G. To every $U \in \mathfrak{g}$ there corresponds a Killing vector ξ_U on X which in addition satisfies that $d\imath_{\xi_U} H = 0$, since H is both closed and invariant. The sigma model term S_σ can be gauged simply by minimal coupling to a gauge field A; that is, to a family of locally defined $\mathfrak{g}$-valued one-forms on Σ. Minimal coupling consists in substituting the exterior derivative for a covariant exterior derivative:

$$d\varphi^i \mapsto d^\nabla \varphi^i := d\varphi^i - A^a \xi_a^i(\varphi) \ , \tag{4}$$

where A^a are the components of A relative to a basis U_a for $\mathfrak{g}$ and ξ_a^i is the i-th component of the Killing vector ξ_{U_a}. The gauged sigma model term is then

$$S_{\mathrm{g}}[\varphi, A] := \tfrac{1}{2}\int_\Sigma g_{ij}(\varphi) d^\nabla \varphi^i \wedge \star d^\nabla \varphi^j = \tfrac{1}{2}\int_\Sigma g_{ij}(\varphi)\nabla_\alpha\varphi^i\nabla_\beta\varphi^j\eta^{\alpha\beta}\,\mathrm{dvol}_\Sigma\ , \tag{5}$$

where we have written $\nabla_\alpha\varphi^i = \partial_\alpha\varphi^i - A_\alpha^a\xi_a^i$.

Gauging the WZ term is a different matter: minimal coupling H does not generally result in a theory with local equations of motion, since the resulting form is not in general closed. Indeed, gauging the WZ term is hindered by a set of obstructions [2, 3, 4] which can be described succinctly in terms of equivariant cohomology [6, 7]. The statement is that the WZ term can be gauged if and only if H extends to an equivariant closed form. Let us explain this statement.

Under an infinitesimal gauge transformation with parameter $\lambda : \Sigma \to \mathfrak{g}$, a differential form $H \in \Omega(X)$ transforms as follows

$$\delta_\lambda H = \lambda^a \mathscr{L}_a H + d\lambda^a \wedge \imath_a H\ , \tag{6}$$

where $\imath_a$ and $\mathscr{L}_a$ are the contraction of and Lie derivative along the Killing vector ξ_{U_a}, respectively, and $\lambda^a : \Sigma \to \mathbb{R}$ are the component functions of λ with respect to the basis U_a. Gauging means coupling to a gauge field A, which is locally a one-form on Σ with values in $\mathfrak{g}$, and which transforms under an infinitesimal gauge transformation as

$$\delta_\lambda A^a = d\lambda^a - f_{bc}{}^a\lambda^b A^c\ .$$

The curvature two-form $F = dA + \frac{1}{2}[A, A]$ transforms as

$$\delta_\lambda F^a = -f_{bc}{}^a\lambda^b F^c\ .$$

It is possible to write these transformations in a way analogous to (6) by introducing the contractions

$$\imath_a A^b = \delta_a{}^b \qquad \text{and} \qquad \imath_a F^b = 0 \tag{7}$$

and the Lie derivatives $\mathscr{L}_a = d\imath_a + \imath_a d$. In this way we have that the infinitesimal gauge transformation of *any* expression involving A and F and differential forms on X is given by (6) from where one reads that such an expression is gauge-invariant provided that it is annihilated by both $\imath_a$ and $\mathscr{L}_a$ for all a. We can formalise this as follows.

Let $\Omega^\bullet(X)$ denote the de Rham complex of differential forms on X. It is a differential graded algebra (dga) relative to the exterior derivative and the wedge product. The G-action on X induces one on $\Omega^\bullet(X)$ and because G is connected, this action is trivial on cohomology. At the level of the Lie algebra, this can be seen as follows. Every element U_a of the Lie algebra defines an antiderivation $\imath_a$ corresponding to contraction with the Killing vector ξ_{U_a}, and also defines a derivation $\mathscr{L}_a = [d, \imath_a]$, which shows that $\mathscr{L}_a$ acts trivially in the cohomology. One has the usual formulae $[\mathscr{L}_a, \imath_b] = f_{ab}{}^c\imath_c$ and $[\mathscr{L}_a, \mathscr{L}_b] = f_{ab}{}^c\mathscr{L}_c$. The existence of these (anti)derivations turns $\Omega^\bullet(X)$ into a G-dga. Now let $(\mathscr{A}, d, \imath)$ be *any* G-dga. We say that an element $\phi \in \mathscr{A}$ is horizontal if $\imath_a\phi = 0$ for all a. Similarly we say that $\phi \in \mathscr{A}$ is invariant if $\mathscr{L}_a\phi = 0$ for all a. If ϕ is both horizontal and invariant we say that it is basic. Note that if ϕ is both horizontal and closed ($d\phi = 0$) then it is automatically invariant and hence basic.

In any G-dga, the basic elements form a subcomplex and hence a differential graded subalgebra. The archetypal example of a G-dga is the Weil algebra $W(\mathfrak{g})$ which is freely generated by an abstract $\mathfrak{g}$-valued one-form A and an abstract $\mathfrak{g}$-valued two-form F subject to the following relations

$$dA = -\tfrac{1}{2}[A, A] + F \qquad \text{and} \qquad dF = [F, A] \tag{8}$$

modelled on the structure equations for a connection on a principal G-bundle. In fact, this is an algebraic model for the de Rham complex of the total space of the universal bundle $EG \to BG$ over the classifying space of the group G. The contractibility of EG is reflected in the fact that $W(\mathfrak{g})$ is acyclic. The antiderivation $\imath_a$ is defined by $\imath_a A = U_a$ and $\imath_a F = 0$, and we define $\mathscr{L}_a = [d, \imath_a]$.

We define $\Omega^\bullet_G(X)$ as the basic subcomplex of the tensor product $W(\mathfrak{g}) \otimes \Omega(X)$, which is naturally a G-dga. The cohomology $H_G(X)$ is called the G-equivariant cohomology of M. As seen above, the basic forms in $W(\mathfrak{g}) \otimes \Omega(X)$ are precisely the gauge-invariant terms in the corresponding sigma model. The gauging of the WZ term consists in extending the WZ term H to a gauge-invariant term which in addition is closed; that is, to an equivariant cocycle. That is, we seek

$$\mathscr{H} = H + \phi_a \wedge A^a + \theta_a \wedge F^a + \tfrac{1}{2}\phi_{ab} \wedge A^a \wedge A^b + \cdots \in \Omega^{d+1}_G(X) \ , \tag{9}$$

where

$$\Omega^{d+1}_G(X) = \bigoplus_{p=0}^{d+1} W^p(\mathfrak{g}) \otimes \Omega^{d+1-p}(X) \ ,$$

such that $d\mathscr{H} = 0$.

The simplest way to derive the explicit expressions for the obstruction to gauging the WZ term is to work with the Cartan model for equivariant cohomology. The departing observation is that the dependence on the gauge field A of a local gauge invariant expression is via the exterior covariant derivative (or F), whence we should be able to dispense with A. The way to do this is to introduce the Cartan model for equivariant cohomology. The basic subcomplex of $W(\mathfrak{g}) \otimes \Omega(X)$ is isomorphic to the complex whose cochains are the G-invariants in $\mathfrak{S}\mathfrak{g}^* \otimes \Omega(X)$ relative to a twisted differential d_C defined by

$$d_C \phi = d\phi - F^a \imath_a \phi \qquad \text{and} \qquad d_C F = 0 \ ,$$

where $\phi \in \Omega(X)$. The map

$$\left(W(\mathfrak{g}) \otimes \Omega(X)\right)_{\text{basic}} \xrightarrow{\cong} \left(\mathfrak{S}\mathfrak{g}^* \otimes \Omega(X)\right)^G$$

is given simply by putting $A = 0$, whereas the inverse map is given by minimal coupling. Therefore we will write the Cartan representative for $\mathscr{H}$ as

$$\mathscr{H}_C = H + \theta_a F^a + \tfrac{1}{2}\theta_{ab} F^a F^b + \cdots$$

with $\mathscr{L}_a \theta_b = f_{ab}{}^c \theta_c$, et cetera, and we demand that $d_C \mathscr{H}_C = 0$, which gives the sequence of conditions

$$\imath_a H = d\theta_a, \qquad \imath_a \theta_b + \imath_b \theta_a = d\theta_{ab}, \qquad \imath_a \theta_{bc} + \text{cyclic} = d\theta_{abc}, \qquad \ldots$$

In each equation the left-hand side is closed and the obstruction to gauging is the obstruction of that closed form being exact.

For example, for a two-dimensional sigma model, which is the case we are primarily interested in, the Cartan representative for the gauged WZ term takes the form

$$\mathscr{H}_C = H + \theta_a F^a \ ,$$

for some 1-forms θ_a satisfying $\mathscr{L}_a \theta_b = f_{ab}{}^c \theta_c$. The obstructions to gauging are given by

$$d_C \mathscr{H}_C = -\imath_a H F^a + d\theta_a F^a - \imath_a \theta_b F^a F^b = 0 \ ,$$

or equivalently

$$\imath_a H = d\theta_a \qquad \text{and} \qquad \imath_a \theta_b = -\imath_b \theta_a \ , \tag{10}$$

which in components become

$$\xi^i_a H_{ijk} = \partial_j \theta_{a\,k} - \partial_k \theta_{a\,j} \qquad \text{and} \qquad \xi^i_a \theta_{b\,i} = -\xi^i_b \theta_{a\,i} \ .$$

Minimally coupling $\mathscr{H}_C$ we obtain

$$\mathscr{H} = H - \imath_a H A^a - \tfrac{1}{2}\imath_a\imath_b H A^a \wedge A^b + \tfrac{1}{6}\imath_a\imath_b\imath_c H A^a \wedge A^b \wedge A^c + \theta_a F^a - \imath_a\theta_b A^a \wedge F^b$$

which, using the relations (10), can be simplified to

$$\mathscr{H} = H + d\left(A^a\theta_a + \tfrac{1}{2}\imath_a\theta_b A^a \wedge A^b\right) \ . \tag{11}$$

As a result, the gauged WZ term is given by

$$\begin{aligned} S_{\mathrm{gWZ}} &= \int_M H + \int_{\varphi(\Sigma)} \left(A^a\theta_a + \tfrac{1}{2}\imath_a\theta_b A^a \wedge A^b\right) \\ &= \int_{\widetilde{\Sigma}} \tfrac{1}{6} H_{ijk}(\varphi)\partial_\alpha\varphi^i\partial_\beta\varphi^j\partial_\gamma\varphi^k\varepsilon^{\alpha\beta\gamma}d^3\sigma \\ &\qquad + \int_\Sigma \left(A^a_\alpha\theta_{a\,i}(\varphi)\partial_\beta\varphi^i + \tfrac{1}{2}\xi^i_a\theta_{b\,i}(\varphi)A^a_\alpha A^b_\beta\right)\varepsilon^{\alpha\beta}d^2\sigma \ , \end{aligned}$$

where $\widetilde{\Sigma}$ is a three-manifold with boundary Σ, to which the φ^i have been extended.

4. Gauging the WZ term with boundary

In this section we present the obstructions to gauging the WZ term in a sigma model with boundary and, for the two-dimensional case, interpret them in terms of Courant brackets.

4.1. **The obstructions from the boundary.** As recalled above, the WZ term of a sigma model with boundary $\varphi : (\Sigma, \partial\Sigma) \to (X, Y)$ is given by

$$\int_M H - \int_D B$$

where $D \subset Y$, $\partial M = \varphi(\Sigma) + D$, $i^*H = dB$ and $i : Y \to X$ is the embedding. Let G act on X in such a way that it preserves the boundary conditions, that is, the submanifold Y. We will further assume that G does not just preserve H but also B. (This means that $i^*[H] = 0$ also in the G-invariant cohomology $H^{d+1}(Y)^G$.) Then we will have gauged this term if we can extend H to a closed equivariant form $\mathscr{H}$ in such a way that $i^*\mathscr{H} = d\mathscr{B}$, with $\mathscr{B}$ an extension of B by terms depending on the gauge field. Equivalently, in the Cartan model, $i^*\mathscr{H}_C = d_C\mathscr{B}_C$. The gauged WZ term is then

$$S_{\mathrm{grWZ}} = \int_M \mathscr{H} - \int_D \mathscr{B} \ .$$

As an illustration let us consider the two-dimensional case. In the Cartan model

$$\mathscr{H}_C = H + \theta_a F^a \qquad \text{and} \qquad \mathscr{B}_C = B + h_a F^a \ ,$$

for some functions h_a and one-forms θ_a satisfying $\mathscr{L}_a h_b = f_{ab}{}^c h_c$ and $\mathscr{L}_a\theta_b = f_{ab}{}^c\theta_c$. As we saw above, H extends to an equivariant closed form if the conditions (10) are satisfied. In addition, the relative condition $i^*\mathscr{H}_C = d_C\mathscr{B}_C$ expands to

$$i^*H = dB \qquad \text{and} \qquad i^*\theta_a = dh_a - \imath_a B \ . \tag{12}$$

Notice that $d(i^*\theta_a + \imath_a B) = 0$ because of invariance of B, and the condition is that this closed form should be exact, that is, dh_a.

Minimally coupling $\mathscr{B}_C$ we obtain

$$\mathscr{B} = B + \imath_a B A^a - \tfrac{1}{2}\imath_a\imath_b B A^a \wedge A^b + h_a F^a \ ,$$

which, using (12), can be simplified to

$$\mathscr{B} = B + i^* \left(A^a\theta_a + \tfrac{1}{2}\imath_a\theta_b A^a \wedge A^b\right) + d(h_a A^a) \ . \tag{13}$$

As a result, the gauged relative WZ term is given by

$$\begin{aligned} S_{\text{grWZ}} &= \int_M H - \int_D B + \int_{\varphi(\Sigma)} \left(A^a\theta_a + \tfrac{1}{2}\imath_a\theta_b A^a \wedge A^b\right) + \int_{\varphi(\partial\Sigma)} h_a A^a \\ &= \int_{\widetilde{\Sigma}} \tfrac{1}{6} H_{ijk}(\varphi)\partial_\alpha\varphi^i\partial_\beta\varphi^j\partial_\gamma\varphi^k\varepsilon^{\alpha\beta\gamma}d^3\sigma \\ &\quad + \int_\Sigma \left(A^a_\alpha\theta_{a\,i}(\varphi)\partial_\beta\varphi^i + \tfrac{1}{2}\xi^i_a\theta_{b\,i}(\varphi)A^a_\alpha A^b_\beta\right)\varepsilon^{\alpha\beta}d^2\sigma \\ &\quad - \int_\Delta \tfrac{1}{2}B_{ij}(\varphi)\partial_\alpha\varphi^i\partial_\beta\varphi^j\varepsilon^{\alpha\beta}d^2\zeta - \int_{\partial\Sigma} h_a(\varphi)A^a_\sigma d\sigma \ , \end{aligned} \tag{14}$$

where now $\widetilde{\Sigma}$ is a 3-manifold with boundary $\partial\widetilde{\Sigma} = \Sigma + \Delta$, and where we have extended the maps φ^i to $\widetilde{\Sigma}$. The local coordinates on Δ are called ζ and the local coordinate on $\partial\Sigma$ is called σ. We notice that, as in the case without boundary, the gauge fields appear algebraically.

4.2. **Gauged WZ terms and Courant brackets.** As observed in [15], there is an interpretation of the conditions (10) for gauging the WZ term of a two-dimensional sigma model in terms of Courant brackets and the generalised geometry of the target X, that is the geometry of $TX \oplus T^*X$. Indeed, if a symmetry group G of the WZ term given by $H \in \Omega^3(X)$ can be gauged, then the image of the map

$$\mathfrak{g} \to C^\infty(TX \oplus T^*X) \qquad \text{defined by} \qquad U_a \mapsto \xi_a + \theta_a$$

is isotropic and involutive under the H-twisted Courant bracket (see, e.g., [16, Section 3.7] and references therein)

$$[v+\alpha, w+\beta]_H := [v,w] + \mathscr{L}_v\beta - \mathscr{L}_w\alpha - \tfrac{1}{2}d(\imath_v\beta - \imath_w\alpha) - \imath_v\imath_w H \ , \tag{15}$$

for all $v+\alpha, w+\beta \in C^\infty(TX \oplus T^*X)$. Indeed,

$$\begin{aligned} [\xi_a+\theta_a, \xi_b+\theta_b]_H &= [\xi_a,\xi_b] + \mathscr{L}_a\theta_b - \mathscr{L}_b\theta_a - \tfrac{1}{2}d(\imath_a\theta_b - \imath_b\theta_a) - \imath_a\imath_b H \\ &= f_{ab}{}^c\xi_c + \tfrac{1}{2}f_{ab}{}^c\theta_c - \tfrac{1}{2}f_{ba}{}^c\theta_c + \tfrac{1}{2}\imath_a d\theta_b - \tfrac{1}{2}\imath_b d\theta_a - \imath_a d\theta_b \\ &= f_{ab}{}^c(\xi_c + \theta_c) \ , \end{aligned}$$

whereas isotropy is simply the condition $\imath_a\theta_b + \imath_b\theta_a = 0$. In other words, there is Lie subalgebroid (isomorphic to $\mathfrak{g}$) of the twisted Courant algebroid on X associated to H. Except for the maximality condition, we might call this a twisted Dirac structure on X.

This result holds also in higher dimensions. Instead of the Courant bracket on $TX \oplus T^*X$ we have to consider its generalisation to $TX \oplus \Lambda^{d-1}T^*X$, also called the Vinogradov bracket (see, e.g., [17]). The expression for this bracket is formally identical to that in (15) except that $\alpha, \beta \in \Omega^{d-1}(X)$. The proof of involutivity is identical to the case of $d = 2$ above. The only point to notice is that $\imath_a\theta_b + \imath_b\theta_a = d\theta_{ab}$, which suggests broadening of the definition of the notion of isotropy, where $\imath_a\theta_b + \imath_b\theta_a$ need not vanish, but merely be exact, which points in the direction of A_∞ structures. Of course for $d = 2$ there is no such freedom since there are no exact 0-forms. This weaker notion of isotropy still ensures the Jacobi identity for involutive sub-bundles. Hence gaugings give rise to Lie subalgebroids (isomorphic to $\mathfrak{g}$) of the twisted Vinogradov algebroid on X associated to H.

The new obstructions due to the presence of the boundary also have a similar interpretation. First of all, the condition that $i^*H = dB$, says that $(Y, B) \subset (X, H)$ is a generalised submanifold in the language of [16, Definition 7.4] and [17, Definition 4]. Now consider the map

$$\mathfrak{g} \to C^\infty(TY \oplus \Lambda^{d-1}T^*Y) \qquad \text{defined by} \qquad U_a \mapsto \xi_a + i^*\theta_a \ ,$$

where we have used that ξ_a are tangent to Y. Let's compute their twisted bracket (on Y):

$$[\xi_a + i^*\theta_a, \xi_b + i^*\theta_b]_{i^*H} \ .$$

Since $i^*H = dB$, we have that

$$[\xi_a + i^*\theta_a, \xi_b + i^*\theta_b]_{i^*H} = [e^B(\xi_a + i^*\theta_a), e^B(\xi_b + i^*\theta_b)] \ ,$$

where the bracket on the right-hand side is the untwisted Courant bracket (simply put $H = 0$ in equation (15)) and where the B-field transform $e^B : C^\infty(TY \oplus \Lambda^{d-1}T^*Y) \to C^\infty(TY \oplus \Lambda^{d-1}T^*Y)$ is defined by

$$e^B(V + \alpha) = V + \alpha + \imath_V B \ .$$

In our case,

$$e^B(\xi_a + i^*\theta_a) = \xi_a + dh_a \ ,$$

where we have used equation (12), which holds also for $d > 2$. Then we have

$$\begin{aligned}
[e^B(\xi_a + i^*\theta_a), e^B(\xi_b + i^*\theta_b)] &= [\xi_a + dh_a, \xi_b + dh_b] \\
&= [\xi_a, \xi_b] + \mathscr{L}_a dh_b - \mathscr{L}_b dh_a - \tfrac{1}{2} d(\imath_a dh_b - \imath_b dh_a) \\
&= [\xi_a, \xi_b] + \mathscr{L}_a dh_b \\
&= f_{ab}{}^c(\xi_c + dh_c) \\
&= f_{ab}{}^c e^B(\xi_c + i^*\theta_c) \ ,
\end{aligned}$$

where we have used that $\mathscr{L}_a d = d\mathscr{L}_a$. Therefore we have again a Lie subalgebroid of the canonical Vinogradov algebroid on Y, which is isomorphic to $\mathfrak{g}$ under the map

$$\mathfrak{g} \to C^\infty(TY \oplus \Lambda^{d-1}T^*Y) \qquad \text{defined by} \qquad U_a \mapsto e^B(\xi_a + i^*\theta_a) \ .$$

5. Gauging the WZW model with boundary

In this section we apply the preceding discussion to the gauging of a WZW model with boundary. We will show that we will be able to gauge any group which can be gauged in the case without boundary, provided that we use boundary conditions which are orbits of the group we are trying to gauge. Two natural classes of such gaugings are (twisted) diagonal subgroups with boundary conditions given by (twisted) conjugacy classes, and chiral isotropic subgroups with boundary conditions given by cosets. We start by setting up the notation for computing with Lie groups.

5.1. **Some yoga about Lie groups.** Throughout this section, G will denote a (connected) Lie group with a bi-invariant metric. We will be interested in gauging subgroups K of the isometry subgroup $G \times G$. Let $\mathfrak{g}$ denote the Lie algebra and let $\langle -, - \rangle$ denote the invariant inner product. Let $\theta_L \in \Omega^1(G; \mathfrak{g})$ denote the left-invariant Maurer–Cartan form on G. It is a $\mathfrak{g}$-valued one-form on G. Identifying $T_eG \cong \mathfrak{g}$, we find that at the identity $\theta_L\big|_e = \mathrm{id}$ and hence at any other point

$$\theta_L\big|_g = \lambda^*_{g^{-1}} \, \mathrm{id} \ ,$$

where $\lambda_g : G \to G$ denotes left-multiplication by g. For matrix groups, $\theta_L\big|_g = g^{-1}dg$ with some abuse of notation. This representation is useful in computations. For example, one can see immediately that θ_L satisfies the structure equation

$$d\theta_L = -\tfrac{1}{2}[\theta_L, \theta_L] \in \Omega^2(G;\mathfrak{g}) \ .$$

The standard three-form $H \in \Omega^3(G)$ can be written as

$$H = \tfrac{1}{6}\left\langle \theta_L, [\theta_L, \theta_L]\right\rangle \ .$$

This shows that H is left-invariant.

Corresponding to every $X \in \mathfrak{g}$ there is a left-invariant vector field X^L obeying $X^L(e) = X$ and hence $X^L(g) = (\lambda_g)_* X$, or equivalently

$$X^L(g) = \left.\frac{d}{dt}\right|_{t=0} \left(g e^{tX}\right) \ .$$

It is clear that $\theta_L(X^L) = X$. This map $X \mapsto X^L$ is an *anti*-homomorphism of Lie algebras, whence $[X^L, Y^L] = -[X, Y]^L$, where the bracket on the left is the Lie bracket of vector fields on G and the one on the right is the one in $\mathfrak{g}$.

Let $j : G \to G$ denote the inverse map: $j(g) = g^{-1}$. Define $\theta_R = j^*\theta_L$; in other words,

$$\begin{aligned}
\theta_R\big|_g &= j^*\theta_L\big|_{g^{-1}} \\
&= j^*\lambda_g^* \,\mathrm{id} \\
&= (\lambda_g \circ j)^* \,\mathrm{id} \\
&= (j \circ \rho_{g^{-1}})^* \,\mathrm{id} \\
&= \rho_{g^{-1}}^* j^* \,\mathrm{id} \ ,
\end{aligned}$$

where we have used that $\lambda_g \circ j = j \circ \rho_{g^{-1}}$, where $\rho_g : G \to G$ denotes right translation by g, sending $g_0 \mapsto g_0 g$. We now notice that at the identity $j_*\big|_e = -\,\mathrm{id}$, whence the same is true for j^*. Therefore

$$\theta_R\big|_g = -\rho_{g^{-1}}^* \,\mathrm{id} \ .$$

Similarly, let $X^R = j_* X^L$. After an analogous calculation to the one above we find that

$$X^R(g) = -(\rho_g)_* X = \left.\frac{d}{dt}\right|_{t=0} \left(e^{-tX} g\right) \ .$$

In particular, $X^R(e) = -X$. Notice that still $\theta_R(X^R) = X$.

Since they are related by pull-backs, θ_R obeys the same structure equation as θ_L:

$$d\theta_R = -\tfrac{1}{2}[\theta_R, \theta_R] \ .$$

In terms of θ_R we find

$$H = -\tfrac{1}{6}\left\langle \theta_R, [\theta_R, \theta_R]\right\rangle \ ,$$

where the sign is due to the fact that at the identity $\theta_R\big|_e = -\theta_L\big|_e$. This expression shows that H is also right-invariant, hence it is bi-invariant.

Finally we collect some useful identities:

$$\theta_R(X^L)\big|_g = -\operatorname{Ad}_g X \qquad \text{and} \qquad \theta_L(X^R)\big|_g = -\operatorname{Ad}_{g^{-1}} X \ , \tag{16}$$

and

$$d\operatorname{Ad}_g X = [\operatorname{Ad}_g X, \theta_R] \qquad \text{and} \qquad d\operatorname{Ad}_{g^{-1}} X = [\operatorname{Ad}_{g^{-1}} X, \theta_L] \ , \tag{17}$$

where $\operatorname{Ad}_g X = (\lambda_g \circ \rho_{g^{-1}})_* X = gXg^{-1}$, with some abuse of notation.

5.2. **The explicit obstructions.** Now let K be a Lie group acting on G by isometries, and hence preserving H, since H is bi-invariant. Since the isometry group of G is $G \times G$, we have a group homomorphism $K \to G \times G$, which in turn is uniquely characterised by two group homomorphisms $\ell, r : K \to G$. At the level of the Lie algebra, we also have homomorphisms $\ell, r : \mathfrak{k} \to \mathfrak{g}$ which determine the map $\mathfrak{k} \to \mathfrak{g} \oplus \mathfrak{g}$.

The action of $G \times G$ on G is given by

$$(g_L, g_R) \cdot g = g_L g g_R^{-1} \ .$$

Therefore if $X \in \mathfrak{k}$, and $(\ell(X), r(X))$ the corresponding element in $\mathfrak{g} \oplus \mathfrak{g}$, then the Killing vector at g is given by

$$\left.\frac{d}{dt}\right|_{t=0} \left(e^{t\ell(X)} g e^{-tr(X)}\right) = \left.\frac{d}{dt}\right|_{t=0} \left(e^{t\ell(X)} g\right) + \left.\frac{d}{dt}\right|_{t=0} \left(g e^{-tr(X)}\right) = -r(X)^L - \ell(X)^R \ .$$

Let us define the Killing vector corresponding to X as

$$\xi_X := r(X)^L + \ell(X)^R \ ,$$

where the sign has been changed so that $[\xi_X, \xi_Y] = \xi_{[X,Y]}$.

Let X_a be a basis of $\mathfrak{k}$ with Killing vectors $\xi_a = r(X_a)^L + \ell(X_a)^R$. And let us calculate $\imath_a H$:

$$\begin{aligned}
\imath_a H &= \tfrac{1}{6} \imath_{r(X_a)^L} \langle \theta_L, [\theta_L, \theta_L] \rangle - \tfrac{1}{6} \imath_{\ell(X_a)^R} \langle \theta_R, [\theta_R, \theta_R] \rangle \\
&= \tfrac{1}{2} \langle r(X_a), [\theta_L, \theta_L] \rangle - \tfrac{1}{2} \langle \ell(X_a), [\theta_R, \theta_R] \rangle \\
&= \langle \ell(X_a), d\theta_R \rangle - \langle r(X_a), d\theta_L \rangle \\
&= d\theta_a \ ,
\end{aligned}$$

where

$$\theta_a = \langle \ell(X_a), \theta_R \rangle - \langle r(X_a), \theta_L \rangle \ .$$

We next check that $\mathscr{L}_a \theta_b = f_{ab}{}^c \theta_c$, for which we first compute

$$\imath_a \theta_b = \langle \ell(X_a), \ell(X_b) \rangle - \langle \ell(X_b), \operatorname{Ad}_g r(X_a) \rangle - \langle r(X_a), r(X_b) \rangle + \left\langle r(X_b), \operatorname{Ad}_{g^{-1}} \ell(X_a) \right\rangle \ ,$$

where we have used equations (16), and

$$\begin{aligned}
\imath_a \imath_b H = &- \langle \ell(X_b), [\ell(X_a), \theta_R] \rangle + \langle \ell(X_b), [\operatorname{Ad}_g r(X_a), \theta_R] \rangle \\
&+ \langle r(X_b), [r(X_a), \theta_L] \rangle - \left\langle r(X_b), [\operatorname{Ad}_{g^{-1}} \ell(X_a), \theta_L] \right\rangle \ .
\end{aligned}$$

Continuing with the calculation, we find, using (17), that

$$\begin{aligned}
\mathscr{L}_a \theta_b &= d\imath_a \theta_b + \imath_a \imath_b H \\
&= - \langle \ell(X_b), d \operatorname{Ad}_g r(X_a) \rangle + \left\langle r(X_b), d \operatorname{Ad}_{g^{-1}} \ell(X_a) \right\rangle + \langle [\ell(X_a), \ell(X_b)], \theta_R \rangle \\
&\quad + \langle \ell(X_b), [\operatorname{Ad}_g r(X_a), \theta_R] \rangle - \langle [r(X_a), r(X_b)], \theta_L \rangle - \left\langle r(X_b), [\operatorname{Ad}_{g^{-1}} \ell(X_a), \theta_L] \right\rangle \\
&= - \langle \ell(X_b), [\operatorname{Ad}_g r(X_a), \theta_R] \rangle + \left\langle r(X_b), [\operatorname{Ad}_{g^{-1}} \ell(X_a), \theta_L] \right\rangle + \langle [\ell(X_a), \ell(X_b)], \theta_R \rangle \\
&\quad + \langle \ell(X_b), [\operatorname{Ad}_g r(X_a), \theta_R] \rangle - \langle [r(X_a), r(X_b)], \theta_L \rangle - \left\langle r(X_b), [\operatorname{Ad}_{g^{-1}} \ell(X_a), \theta_L] \right\rangle \\
&= \langle [\ell(X_a), \ell(X_b)], \theta_R \rangle - \langle [r(X_a), r(X_b)], \theta_L \rangle \ ,
\end{aligned}$$

which yields the desired result after using that ℓ and r are Lie algebra homomorphisms.

Finally the obstruction $\imath_a\theta_b + \imath_b\theta_a = 0$ translates into

$$\langle \ell(X_a), \ell(X_b)\rangle = \langle r(X_a), r(X_b)\rangle \ . \tag{18}$$

In other words, the scalar products induced on $\mathfrak{k}$ via $\ell, r : \mathfrak{k} \to \mathfrak{g}$ should balance.

There are (at least) two natural ways in which this can be achieved: one is by a diagonal gauging in which $\ell = r$, or more generally a twisted diagonal gauging in which $\ell = r \circ \tau$, for some orthogonal automorphism τ of $\mathfrak{g}$. The other general class are chiral gaugings where, say, $r = 0$ and where $\ell(\mathfrak{k}) \subset \mathfrak{g}$ is an isotropic subalgebra. We will discuss both cases at the end of this section, once we have shown that any group for which the WZW model without boundary can be gauged, can also be gauged when the boundary is present, provided that we choose appropriately symmetric boundary conditions.

5.3. **The boundary offers no new obstructions.** Let us consider gauging a group K acting via isometries on G. As discussed above this action is defined by group homomorphisms $\ell, r : K \to G$. The action of $k \in K$ on G is denoted

$$\mathrm{Ad}_k^{\ell,r} : G \to G \qquad \text{where} \qquad \mathrm{Ad}_k^{\ell,r}(g) = \ell(k) g r(k)^{-1} \ .$$

The possible boundary conditions are the orbits of this action:

$$C^{\ell,r}(g_0) = \left\{ \ell(k) g_0 r(k)^{-1} \mid k \in K \right\} \ .$$

Fix $g_0 \in G$ and let $Y := C^{\ell,r}(g_0)$. We will denote $i : Y \to G$ the canonical embedding.

We will first show that $i^*H = dB$ for some two-form $B \in \Omega^2(Y)$. To show this and for later usage, it is convenient to write down the pull-backs of the Maurer–Cartan forms to Y. Letting $g = \mathrm{Ad}_k^{\ell,r} g_0$ and letting $\vartheta = dkk^{-1}$, we find

$$i^*\theta_L\big|_g = (\mathrm{Ad}_{g^{-1}} \ell - r)\vartheta \qquad \text{and} \qquad i^*\theta_R\big|_g = (\mathrm{Ad}_g r - \ell)\vartheta \ , \tag{19}$$

where we let ℓ, r also denote the Lie algebra homomorphisms $\mathfrak{k} \to \mathfrak{g}$.

A short calculation, using equation (18), shows that

$$i^*H\big|_g = \langle \alpha\vartheta, rd\vartheta\rangle - \langle r\vartheta, \alpha d\vartheta\rangle \ ,$$

where we have introduced the notation $\alpha = \mathrm{Ad}_{g^{-1}} \ell$. Now consider

$$B = -\langle r\vartheta, \alpha\vartheta\rangle = -\langle r\vartheta, i^*\theta_L\rangle \ .$$

Differentiating, one gets

$$\begin{aligned} dB &= -\langle rd\vartheta, i^*\theta_L\rangle + \langle r\vartheta, i^*d\theta_L\rangle \\ &= -\langle rd\vartheta, i^*\theta_L\rangle - \tfrac{1}{2}\langle \vartheta, [i^*\theta_L, i^*\theta_L]\rangle \ . \end{aligned}$$

We now use the first equation in (19), expand and use that r and α are Lie algebra homomorphisms, to obtain that

$$dB = \langle rd\vartheta, \alpha\vartheta\rangle - \langle r\vartheta, \alpha d\vartheta\rangle = i^*H \ .$$

We now show that B is K-invariant. Indeed, if $x \in K$, then $\mathrm{Ad}_x^{\ell,r} g = \ell(xk) g_0 r(xk)^{-1}$, whence k changes by left multiplication by x, so that in turn ϑ changes to $\mathrm{Ad}_x \vartheta$. Finally we calculate

$$\begin{aligned} \left(\mathrm{Ad}_x^{\ell,r}\right)^* B_{\mathrm{Ad}_x^{\ell,r} g} &= -\left\langle r\left(\mathrm{Ad}_x^{\ell,r}\right)^*\vartheta, \mathrm{Ad}_{(\ell(x)gr(x)^{-1})^{-1}} \ell \left(\mathrm{Ad}_x^{\ell,r}\right)^*\vartheta\right\rangle \\ &= -\left\langle r \,\mathrm{Ad}_x \vartheta, \mathrm{Ad}_{r(x)} \mathrm{Ad}_{g^{-1}} \mathrm{Ad}_{\ell(x)^{-1}} \ell \,\mathrm{Ad}_x \vartheta\right\rangle \\ &= -\left\langle \mathrm{Ad}_{r(x)^{-1}} r \,\mathrm{Ad}_x \vartheta, \mathrm{Ad}_{g^{-1}} \mathrm{Ad}_{\ell(x)^{-1}} \ell \,\mathrm{Ad}_x \vartheta\right\rangle \\ &= B_g \ , \end{aligned}$$

where we have used that the metric is Ad-invariant and that for any homomorphism τ, whence in particular for ℓ and r,

$$\mathrm{Ad}_{\tau(x)^{-1}} \tau \,\mathrm{Ad}_x = \tau \ ,$$

where the Ad in the left is in K and the one in the right is in G. This remark also applies to the above calculation, where we hope no confusion should result as a consequence of this ambiguity in the notation.

This means that the relative WZ term with these boundary conditions is invariant under the (global) action of K and we can address the problem of gauging it. We will prove that we can do so.

Given $X_a \in \mathfrak{k}$, the corresponding Killing vector is

$$\xi_a = \ell(X_a)^R + r(X_a)^L \ ,$$

whence $\imath_a H = d\theta_a$, where

$$\theta_a = \langle \ell(X_a), \theta_R \rangle - \langle r(X_a), \theta_L \rangle \ .$$

Pulling back θ_a to Y we find

$$\begin{aligned} i^*\theta_a\big|_g &= \langle \ell(X_a), i^*\theta_R \rangle - \langle r(X_a), i^*\theta_L \rangle \\ &= \langle \ell(X_a), (\mathrm{Ad}_g r - \ell)\vartheta \rangle - \left\langle r(X_a), (\mathrm{Ad}_{g^{-1}} \ell - r)\vartheta \right\rangle \\ &= \langle \alpha(X_a), r\vartheta \rangle - \langle r(X_a), \alpha\vartheta \rangle \ , \end{aligned}$$

where we have again used equation (18).

On the other hand, in the chosen parametrisation for $g = \mathrm{Ad}_k^{\ell,r} g_0$, ξ_a generates

$$k \mapsto e^{-tX_a} k$$

whence it corresponds to X_a^R. Since ϑ pulls back to $-\theta_R$ on K, we see that $\imath_a\vartheta = -X_a$. Using this we find,

$$\begin{aligned} \imath_a B &= -\langle r\imath_a\vartheta, \alpha\vartheta \rangle + \langle r\vartheta, \alpha\imath_a\vartheta \rangle \\ &= \langle r(X_a), \alpha\vartheta \rangle - \langle \alpha X_a, r\vartheta \rangle \\ &= -i^*\theta_a \ . \end{aligned}$$

From equation (12), we see that $dh_a = 0$, whence h_a defines an element $h \in \mathfrak{k}^*$. The equivariance condition $\mathscr{L}_a h_b = f_{ab}{}^c h_c$ means that h lies in the annihilator $[\mathfrak{k}, \mathfrak{k}]^o$ of the first derived ideal.

Since $dh_a = 0$, the boundary Lie algebroid in this case is the canonical Lie algebroid associated to the action of $\mathfrak{g}$ on Y.

In summary, we have proved that any subgroup which can be gauged in the WZW model without boundary, can still be gauged in the presence of a boundary provided that the boundary conditions are chosen appropriately, namely they are invariant under the group we are trying to gauge, or more geometrically, they consist of orbits of the action.

It is now a simple matter to write down the gauged WZ term by specialising equation (14) to the present case. Doing so we obtain

$$
\begin{aligned}
S_{\mathrm{grWZW}} = \int_{\widetilde{\Sigma}} \tfrac{1}{3}\operatorname{Tr}(g^{-1}dg)^3 &+ \int_{\Sigma} \operatorname{Tr}\left(g^{-1}\ell(A_z)gr(A_{\bar z}) - g^{-1}\ell(A_{\bar z})gr(A_z)\right) d^2z\\
&- \int_{\Sigma} \operatorname{Tr}\left(\ell(A_z)\partial_{\bar z}gg^{-1} - \ell(A_{\bar z})\partial_z gg^{-1} + r(A_z)g^{-1}\partial_{\bar z}g - r(A_{\bar z})g^{-1}\partial_z g\right) d^2z\\
- \int_{\Delta} \operatorname{Tr}&\left(g_0 r(k)^{-1}\partial_\zeta r(k) g_0^{-1}\ell(k)^{-1}\partial_{\bar\zeta}\ell(k) - g_0 r(k)^{-1}\partial_{\bar\zeta} r(k) g_0^{-1}\ell(k)^{-1}\partial_\zeta\ell(k)\right) d^2\zeta\\
&+ \int_{\partial\Sigma} h(A_x)dx \ , \qquad (20)
\end{aligned}
$$

where again $\widetilde{\Sigma}$ is a three-manifold with boundary $\partial\widetilde{\Sigma} = \Sigma + \Delta$, where z is a local complex coordinate on Σ, ζ a local complex coordinate on Δ and x is a local real coordinate on $\partial\Sigma$. In this formula we have used Tr to mean the invariant scalar product in the Lie algebra. This inner product encodes the information on the "level" of the WZW model. The symbol g denotes both the map $g : \Sigma \to G$ as well as its extension to $\widetilde{\Sigma}$, whereas in Δ, we parametrise $g = \ell(k)g_0 r(k)^{-1}$ in terms of a map $k : \Delta \to K$. The last two terms in equation (20) are due to the presence of the boundary.

5.4. **Two examples.** Now we consider the two natural examples of gaugings.

5.4.1. *(Twisted) Diagonal gaugings.* First we have G embedded diagonally in $G \times G$: $\ell = r = \mathrm{id}$ or more generally as the graph of an orthogonal automorphism $\tau : G \to G$. The possible boundary conditions are the orbits of this action which, in the general case of the graph of an automorphism are twisted conjugacy classes

$$
C^\tau(g_0) = \left\{\tau(g)g_0g^{-1} \mid g \in G\right\} \ .
$$

Such orbits are well-known in the context of D-branes on Lie groups [18, 19, 20].

This corresponds to $\ell = \tau$ and $r = \mathrm{id}$ in the previous section and the results from that section can be immediately imported. We have $i^*H = dB$, where [21] (see also [18, 10] for the untwisted case)

$$
B = -\langle\vartheta, \alpha\vartheta\rangle \ ,
$$

where now $\alpha = \mathrm{Ad}_{g^{-1}}\tau$. As we showed above, B is invariant under the twisted adjoint action, meaning that the relative WZ term with such boundary conditions is invariant under the global action of G via twisted conjugation. As shown above in a more general context, this global action can be gauged.

The action can be read off from equation (20) by putting $r = \mathrm{id}$ and $\ell = \tau$. The new terms due to the boundary are now

$$
\begin{aligned}
S_{\mathrm{grWZW}} = \cdots - \int_{\Delta} \operatorname{Tr}\left(g_0k^{-1}\partial_\zeta k g_0^{-1}\tau(k)^{-1}\partial_{\bar\zeta}\tau(k) - g_0k^{-1}\partial_{\bar\zeta}kg_0^{-1}\tau(k)^{-1}\partial_\zeta\tau(k)\right) d^2\zeta\\
+ \int_{\partial\Sigma} h(A_x)dx \ ,
\end{aligned}
$$

The boundary piece in the gauged action depends on an element $h \in [\mathfrak{g},\mathfrak{g}]^o$. Using the invariant scalar product, h defines an element in the centre of $\mathfrak{g}$. In other words, the gauge field $h(A_x)$ on the boundary takes values in the centre of the Lie algebra, in particular it is always abelian. Semisimple Lie algebras have no centre, which explains why this term in absent in [12, 13, 14].

5.4.2. *Chiral isotropic gaugings.* Another set of subgroups which can be gauged are chiral subgroups where, say, $r = 0$. In this case, the condition to be able to gauge the WZ term (without boundary) is that

$$\langle \ell(X_a), \ell(X_b) \rangle = 0 \ .$$

One class of such subgroups is the following. Let $\mathfrak{g}$ be a maximally noncompact real form of a complex simple Lie algebra and let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$$

be a maximal toral decomposition, where $\mathfrak{h}$ is the Cartan subalgebra and

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Phi_\pm} \mathfrak{g}_\alpha \ ,$$

where $\Phi = \Phi_+ \sqcup \Phi_-$ is the root system, with Φ_+ (resp. Φ_-) the positive (resp. negative) roots, and $\mathfrak{g}_\alpha$ the one-dimensional root space with root α. For example, we can take $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ to be the Lie algebra of traceless $n \times n$ matrices with real entries, $\mathfrak{h}$ the subalgebra of diagonal matrices, and $\mathfrak{n}_\pm$ the subalgebras of strictly triangular matrices. We let $N_\pm \subset G$ denote the subgroup with Lie algebra $\mathfrak{n}_\pm$.

The subalgebra $\mathfrak{n}_+$ is isotropic under the Killing form and we can try to gauge the corresponding group as a chiral gauging: $\mathfrak{n}_+ \to \mathfrak{g} \oplus \mathfrak{g}$ sending $X \to (X, 0)$, for example. The possible boundary conditions are orbits of N_+ under this action, which here correspond to left cosets

$$Y := N_+ g_0 = \{ h g_0 \mid h \in N_+ \} \ .$$

Let $i : Y \to G$ denote the embedding. As we saw above, we can gauge such a symmetry with these boundary conditions. It is perhaps instructive to redo the calculation in this case, since as we will see the two-form B can be taken to be zero. As a result the boundary WZW model is unchanged and the gauged model only receives a contribution from the boundary of the world-sheet.

Let $g = h g_0$ with $h \in N_+$. Then we find

$$i^* \theta_L \big|_g = \mathrm{Ad}_{g^{-1}} \vartheta \qquad \text{and} \qquad i^* \theta_R \big|_g = -\vartheta \ ,$$

where $\vartheta = dh h^{-1}$ as before. Then

$$i^* H = \tfrac{1}{2} \langle \vartheta, [\vartheta, \vartheta] \rangle = 0 \ ,$$

since ϑ is $\mathfrak{n}_+$-valued, and this subalgebra is isotropic. This means that we can take $B = 0$.

If X_a is a basis for $\mathfrak{n}_+$, the corresponding Killing vector is X_a^R. Then

$$\begin{aligned} \imath_a H &= -\tfrac{1}{2} \langle X_a, [\theta_R, \theta_R] \rangle \\ &= \langle X_a, d\theta_R \rangle \\ &= d \langle X_a, \theta_R \rangle \ , \end{aligned}$$

whence $\theta_a = \langle X_a, \theta_R \rangle$. Pulling back to Y, we find

$$i^* \theta_a = \langle X_a, i^* \theta_R \rangle = - \langle X_a, \vartheta \rangle = 0$$

again by isotropy of $\mathfrak{n}_+$.

Therefore $\imath_a B + i^* \theta_a = 0$ identically and we again have that $dh_a = 0$, whence it defines an element $h \in \mathfrak{n}_+^*$ which annihilates $[\mathfrak{n}_+, \mathfrak{n}_+]$. In other words, it necessarily annihilates all the non-simple roots. Hence the gauge field on the boundary is a linear combination $h_i A^i$ where i runs through the simple roots.

The gauged WZ term can be read off from equation (20) and we obtain

$$S_{\mathrm{grWZW}} = \int_{\widetilde{\Sigma}} \tfrac{1}{3}\operatorname{Tr}(g^{-1}dg)^3 - \int_{\Sigma} \operatorname{Tr}\left(\ell(A_z)\partial_{\bar z} g g^{-1} - \ell(A_{\bar z})\partial_z g g^{-1}\right) d^2z + \int_{\partial\Sigma} h(A_x)dx \;.$$

These chiral gaugings are well-known in the context of integrable models as they provide a WZW realisation of the Drinfel'd–Sokolov reduction. The gauged WZW models with boundary could play a role in the construction of boundary integrable models and we hope to return to this topic elsewhere.

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Appendix A. G-equivariant relative cohomology

Let $i : Y \to X$ be a submanifold and let G be a connected Lie group acting on X preserving Y, so that i is G-equivariant. With this data we can define at least two notions of "G-equivariant cohomology of X relative to Y" depending on which model we choose for the relative de Rham complex. In the absence of further assumptions — e.g., compactness of the group — both choices give different theories, and only one of them is relevant to the physical problem at hand. In this Appendix we try to explain this situation.

Recall that there are two complexes computing the de Rham cohomology of X relative to Y. Let $i^* : \Omega(X) \to \Omega(Y)$ denote the operation of pulling back forms from X to Y and let $\Omega(X,Y)$ denote its kernel. We thus have a short exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet(X,Y) \longrightarrow \Omega^\bullet(X) \xrightarrow{\;i^*\;} \Omega^\bullet(Y) \longrightarrow 0 \;. \tag{21}$$

Exactness at the right is simply the possibility to extend a form on Y smoothly to all of X. This short exact sequence gives a long exact sequence in cohomology whose coboundary map $H^p(Y) \to H^{p+1}(X,Y)$ is obtained by sending $[\omega] \in H^p(Y)$ to $[d\widetilde{\omega}]$, where $\widetilde{\omega}$ is an extension of ω to X. The cohomology space $H^*(Y,X)$ is the de Rham cohomology of X relative to Y.

One can relax the notion that a relative form should vanish when pulled back to Y to the level of cohomology alone; that is, that a relative cocycle need not vanish when pulled back to Y but that it should be exact there. This suggests defining a different complex

$$\widetilde{\Omega}^p(X,Y) := \Omega^p(X) \oplus \Omega^{p-1}(Y)$$

with differential $\widetilde{d} : \widetilde{\Omega}^p(X,Y) \to \widetilde{\Omega}^{p+1}(X,Y)$ defined by

$$\widetilde{d}(\omega,\theta) = (d\omega, d\theta + (-1)^p i^*\omega) \ .$$

A cocycle is then a closed form ω whose pull-back to Y is exact.

This complex also fits in a short exact sequence

$$0 \longrightarrow \Omega^{\bullet-1}(Y) \longrightarrow \widetilde{\Omega}^\bullet(X,Y) \longrightarrow \Omega^\bullet(X) \longrightarrow 0 \ , \tag{22}$$

where the first map is simply the inclusion into the second factor $\theta \mapsto (0,\theta)$ and the second map is the projection onto the first $(\omega,\theta) \mapsto \omega$. It is possible to match the resulting long exact sequence in cohomology with the one coming from the first complex and in this way show that the map $\Omega(X,Y) \to \widetilde{\Omega}(X,Y)$ defined by $\omega \mapsto (\omega,0)$ is a quasi-isomorphism, whence the two complexes compute the same cohomology.

Now suppose that G acts on X and Y with $i : Y \to X$ G-equivariant. Then both relative de Rham complexes become G-dgas in natural ways. First of all, since G acts on Y, the Killing vectors are tangent to Y when restricted to Y, whence the derivations $\imath_a$ and $\mathscr{L}_a$ restrict to $\Omega(X,Y)$. Similarly for the second relative de Rham complex $\widetilde{\Omega}(X,Y)$, we define $\imath_a(\omega,\theta) = (\imath_a\omega, \imath_a\theta)$ and $\mathscr{L}_a(\omega,\theta) = (\mathscr{L}_a\omega, \mathscr{L}_a\theta)$. One checks that again $\mathscr{L}_a = \widetilde{d}\imath_a + \imath_a\widetilde{d}$. Moreover the quasi-isomorphism $\Omega(X,Y) \to \widetilde{\Omega}(X,Y)$ is a morphism of G-dgas. This means, for example, that we get a map between the cohomologies of the invariant complexes $H(X,Y)^G \to \widetilde{H}(X,Y)^G$, which however need not be an isomorphism. Indeed, trying to prove the isomorphism as for the case of the relative de Rham complexes, we come across the fact that whereas we still have a short exact sequence of invariant complexes

$$0 \longrightarrow \Omega^{\bullet-1}(Y)^G \longrightarrow \widetilde{\Omega}^\bullet(X,Y)^G \longrightarrow \Omega^\bullet(X)^G \longrightarrow 0 \ , \tag{23}$$

the other sequence

$$0 \longrightarrow \Omega^\bullet(X,Y)^G \longrightarrow \Omega^\bullet(X)^G \xrightarrow{\ i^*\ } \Omega^\bullet(Y)^G \tag{24}$$

now fails to be right-exact in general, because an invariant form on Y need not extend to an *invariant* form on all of X. This will be the case if G is compact, by averaging, for example, but is not the case for general G.

This means that we have two possible definitions of G-equivariant cohomology of X relative to Y, depending on which complex we take for relative cohomology. On the one hand we have

$$\Omega_G(X,Y) := (W(\mathfrak{g}) \otimes \Omega(X,Y))_{\text{basic}}$$

which is a subcomplex of $\Omega_G(X)$. On the other hand, we have

$$\widetilde{\Omega}_G(X,Y) := \Big(W(\mathfrak{g}) \otimes \widetilde{\Omega}(X,Y)\Big)_{\text{basic}} \ .$$

In general both complexes compute different cohomologies; although it is clear that it is this latter complex which is relevant for our problem.

One might worry about another source of ambiguity coming from the order of operations; that is, whether we first "relativise" or "equivariantise" the complex. Starting with the G-dga $\Omega(X)$, one can first consider its equivariant cohomology and then make this relative to Y, or one can first consider the relative cohomology and then make this equivariant. It is not hard to see, however, that for either of the two models of the relative de Rham complex considered above, both procedures yield isomorphic complexes.

Having identified the right equivariant relative de Rham complex, it should be possible, as in [6, 7], to derive a priori vanishing theorems which guarantee the absence of obstructions to gauging for certain types of groups and certain geometries.

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